Remarks on the motion of non-closed planar curves governed by shortening-straightening flow (Nonlinear evolution equations and related topics to mathematical analysis of a phenomena)

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Remarks on the motion of non-closed planar curves governed by shortening-straightening flow

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Abstract

We consider a motion of non-closed planar curves with infinite length. The motion is governed by a steepest descent flow for the geometric functional which consists of length functional and total squared curvature. We call the flow the shortening-straightening flow. In this note, we announce a result of long time existence of shortening-straightening flow for non-closed planar curves with infinite length.

1 Introduction

In this note, we announce a result given in [9]. Let γ be a planar curve, κ be the curvature, and s denote the arc-length parameter of γ. For γ, we consider the following geometric functional

$$E(\gamma)=\lambda^2\int_\gamma ds+\int_\gamma \kappa^2\,ds, \tag{1.1}$$

where λ is a given constant. The steepest descent flow for (1.1) is given by the system

$$\partial_t\gamma=(-2\partial_s^2\kappa-\kappa^3+\lambda^2\kappa)\boldsymbol{\nu}, \tag{1.2}$$

where $\boldsymbol{\nu}$ is the unit normal vector of the curve pointing in the direction of the curvature. Removing λ^2 from the first term of $E(\gamma)$, the term denotes the length functional of γ. We call the steepest descent flow for length functional the curve shortening flow. On the other hand, the second term in (1.1) is well known as the total squared curvature. The steepest descent flow for the functional is called the curve straightening flow. Thus we call (1.2) the shortening-straightening flow in this note.

There are various studies about the steepest descent flow for geometric functional defined on closed curves, for example, the shortening flow ([1], [4], [5]), the straightening flow for curve with fixed total length ([7], [11], [12]), and the straightening flow for curve with fixed local length ([6], [8]). We mention the known results of shortening-straightening flow. In 1996, it has been proved by A. Polden ([10]) that the equation (1.2) admits smooth solutions globally defined in time, when the initial curve is closed and has finite length (i.e., compact without boundary). Furthermore, G. Dziuk, E. Kuwert, and R. Schätzle ([3]) extended the Polden's result of [10] to closed curves with finite length in $\mathbb{R}^n$.

We are interested in the following problem: "What is the dynamics of *non-closed* planar curves with *infinite* length governed by shortening-straightening flow?" In this note, we prove that there exists a long time solution of shortening-straightening flow starting from smooth planar curve with infinite length. Namely, we consider the following initial-boundary value problem:

$$\text{(SS)} \qquad \begin{cases} \partial_t \gamma = (-2\partial_s^2 \kappa - \kappa^3 + \lambda^2 \kappa)\nu, \\ \gamma(0,t) = (0,0), \quad \kappa(0,t) = 0, \\ \gamma(x,0) = \gamma_0(x), \end{cases}$$

where let $\gamma_0(x) = (\phi(x), \psi(x)) : [0,\infty) \to \mathbb{R}^2$ satisfy the following conditions:

$$\text{(C1)} \qquad \gamma_0(0) = (0,0), \quad \kappa_0(0) = 0,$$

$$\text{(C2)} \qquad |\gamma_0{}'(x)| \equiv 1,$$

$$\text{(C3)} \qquad \partial_x^m \kappa_0 \in L^2(0,\infty) \quad \text{for all } m \geq 0,$$

$$\text{(C4)} \qquad \lim_{x\to\infty} \phi(x) = \infty, \quad \lim_{x\to\infty} \phi'(x) = 1,$$

$$\text{(C5)} \qquad \psi(x) = O(x^{-\alpha}) \text{ for some } \alpha > \frac{1}{2} \text{ as } x \to \infty, \quad \psi' \in L^2(0,\infty),$$

where κ_0 is the curvature of γ_0. The condition (C2) and definition of γ_0 imply that γ_0 has infinite length. Moreover, from the conditions (C1)–(C5), we see that γ_0 starts from the origin and is allowed to have self-intersections, but must be close to an axis in C^1 sense as $x \to \infty$. Then the main result of this note is stated as follows:

Theorem 1.1. ([9]) *Let $\gamma_0(x)$ be a planar curve satisfying* (C1)–(C5). *Then there exists a smooth curve $\gamma_\infty(x,t) : [0,\infty) \times [0,\infty) \to \mathbb{R}^2$ satisfying* (SS) *on $[0,R) \times [0,R)$ for sufficiently large R.*

We construct a solution of (SS) by making use of the Arzela-Ascoli theorem. To define an approximate sequence, we solve (1.2) for a certain compact case with fixed boundary (see Section 2).

The paper is organized as follows: In Section 2, we prove that, for planar curves with finite length, there exists a unique classical solution of (1.2) under certain boundary conditions. Making use of result obtained in Section 2, we prove a existence of solution of (SS) in Section 3.

2 Compact case with fixed boundary

In this section, we consider (1.2) for planar curves with finite length under certain boundary conditions. Let $\Gamma_0(x) : [0,L] \to \mathbb{R}^2$ be a smooth planar curve and $k_0(x)$ denote the curvature. Let $\Gamma_0(x)$ satisfy

$$\text{(2.1)} \qquad |\Gamma_0{}'(x)| \equiv 1, \quad \Gamma_0(0) = (0,0), \quad \Gamma_0(L) = (R,0), \quad k_0(0) = k_0(L) = 0,$$

where $L>0$ and $R>0$ are given constants. In this section, let us consider the following initial boundary value problem:

$$\text{(CSS)} \qquad \begin{cases} \partial_t\gamma = (-2\partial_s^2\kappa - \kappa^3 + \lambda^2\kappa)\nu, \\ \gamma(0,t) = (0,0), \quad \gamma(L,t) = (R,0), \quad \kappa(0,t) = \kappa(L,t) = 0, \\ \gamma(x,0) = \Gamma_0(x). \end{cases}$$

The purpose of this section is to prove a long time existence of solution of (CSS).

2.1 Short time existence

First we show a short time existence of solution to (CSS). For this purpose, let

$$\gamma(x,t) = \Gamma_0(x) + d(x,t)\nu_0(x), \tag{2.2}$$

where $d(x,t) : [0,L]\times[0,\infty) \to \mathbb{R}$ is unknown scalar function and $\nu_0(x)$ is the unit normal vector of $\Gamma_0(x)$, i.e., $\nu_0(x) = \Re\Gamma_0{}'(x) = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\Gamma_0{}'(x)$. Under the formulation (2.2), the boundary condition $\gamma(0,t) = (0,0)$ and $\gamma(L,t) = (R,0)$ is reduced to

$$d(0,t) = d(L,t) = 0. \tag{2.3}$$

With the aid of Frenet-Serret's formula $\Gamma_0{}'' = k_0\nu_0$ and $\nu_0{}' = -k_0\Gamma_0{}'$, we have

$$\begin{aligned} \partial_x\gamma &= (1-k_0d)\Gamma_0{}' + \partial_x d\nu_0, \\ \Re\partial_x\gamma &= -\partial_x d\Gamma_0{}' + (1-k_0d)\nu_0, \\ \partial_x^2\gamma &= (-k_0{}'d - 2k_0\partial_x d)\Gamma_0{}' + (\partial_x^2 d + k_0 - k_0{}^2 d)\nu_0, \\ \kappa &= \frac{\partial_x^2\gamma\cdot\Re\partial_x\gamma}{|\partial_x\gamma|^3} = \frac{\partial_x d(k_0{}'d + 2k_0\partial_x d) + (1-k_0d)(\partial_x^2 d + k_0 - k_0{}^2 d)}{\{(1-k_0d)^2 + (\partial_x d)^2\}^{3/2}}. \end{aligned}$$

Thus the condition $\kappa(0,t) = \kappa(L,t) = 0$ is equivalent to

$$\partial_x^2 d(0,t) = \partial_x^2 d(L,t) = 0. \tag{2.4}$$

Since

$$s(x,t) = \int_0^x |\partial_x\gamma(x,t)|\,dx = \int_0^x \{(1-k_0(x)d(x,t))^2 + (\partial_x d(x,t))^2\}^{1/2}\,dx,$$

we have

$$\frac{\partial s}{\partial x} = \{(1-k_0(x)d(x,t))^2 + (\partial_x d(x,t))^2\}^{1/2} := |\gamma_d|. \tag{2.5}$$

Combining the relation (2.5) with

$$\frac{\partial}{\partial s} = \frac{\partial/\partial x}{\partial s/\partial x},$$

we obtain

$$\frac{\partial}{\partial s} = \frac{\partial_x}{|\gamma_d|}.$$

Then we see that

$$\partial_s^2 \kappa = \frac{\partial_x}{|\gamma_d|}\left(\frac{\partial_x}{|\gamma_d|}\left(\frac{\partial_x d(\partial_x k_0 d + 2k_0 \partial_x d) + (1-k_0 d)(\partial_x^2 d + k_0 - k_0{}^2 d)}{|\gamma_d|^3}\right)\right).$$

This is reduced to

$$\partial_s^2 \kappa = \frac{1}{|\gamma_d|^5}\partial_x^2 \alpha_3 - \frac{7}{|\gamma_d|^6}\partial_x |\gamma_d|\, \partial_x \alpha_3 + \left\{-\frac{3}{|\gamma_d|^6}\partial_x^2 |\gamma_d| + \frac{15}{|\gamma_d|^6}(\partial_x |\gamma_d|)^2\right\}\alpha_3,$$

where

$$\alpha_3 = \partial_x d(\partial_x k_0 d + 2k_0 \partial_x d) + (1-k_0 d)(\partial_x^2 d + k_0 - k_0{}^2 d).$$

Setting

$$\begin{aligned}
\alpha_1 &= \partial_x k_0 d + k_0 \partial_x d, \\
\alpha_2 &= \partial_x d \partial_x^2 d + \alpha_1 (k_0 d - 1), \\
\alpha_4 &= \partial_x d \partial_x^3 d + (\partial_x^2 d)^2 + \alpha_1{}^2 + \partial_x \alpha_1 (k_0 d - 1),
\end{aligned}$$

we have

$$\partial_x |\gamma_d| = \frac{\alpha_2}{|\gamma_d|}, \qquad \partial_x^2 |\gamma_d| = -\frac{\alpha_2{}^2}{|\gamma_d|^3} + \frac{\alpha_4}{|\gamma_d|}.$$

Thus $\partial_s^2 \kappa$ is written as

$$\partial_s^2 \kappa = \frac{1}{|\gamma_d|^5}\partial_x^2 \alpha_3 - \frac{1}{|\gamma_d|^7}(7\alpha_2 \partial_x \alpha_3 - 3\alpha_3 \alpha_4) + \frac{18}{|\gamma_d|^9}\alpha_2{}^2 \alpha_3.$$

Since $\kappa = \alpha_3 / |\gamma_d|^3$ and

$$\partial_t \gamma = \partial_t d \nu_0,$$

we have

$$\begin{aligned}
\partial_t d &= \left\{-\frac{2}{|\gamma_d|^4}\partial_x^2 \alpha_3 + \frac{14}{|\gamma_d|^6}\alpha_2 \partial_x \alpha_3 + \frac{6}{|\gamma_d|^6}\alpha_3 \alpha_4 - \frac{36}{|\gamma_d|^8}\alpha_2{}^2 \alpha_3 - \frac{\alpha_3{}^3}{|\gamma_d|^8} + \frac{\lambda^2 \alpha_3}{|\gamma_d|^2}\right\}\frac{1}{1-k_0 d} \\
&= -\frac{2}{|\gamma_d|^4}\partial_x^4 d + \Phi(d).
\end{aligned}$$

Setting $A(d) = (-2/|\gamma_d|^4)\partial_x^4$, the problem (CSS) is written in terms of d as follows:

$$\left\{\begin{aligned}
&\partial_t d = A(d) d + \Phi(d), \\
&d(0,t) = d(L,t) = d''(0,t) = d''(L,t) = 0, \\
&d(x,0) = d_0(x) = 0.
\end{aligned}\right. \tag{2.6}$$

We find a smooth solution of (2.6) for a short time. To do so, we need to show the operator $A(d_0)$ is sectorial. Since $A(d_0) = -2\partial_x^4$, first we consider the boundary value problem

$$
\begin{cases} \partial_x^4 \varphi + \mu \varphi = f, \\ \varphi(0) = \varphi(L) = \varphi''(0) = \varphi''(L) = 0, \end{cases} \tag{2.7}
$$

where μ is a constant. The solution of (2.7) is written as

$$
\varphi(x) = \int_0^L G(x,\xi) f(\xi)\, d\xi, \tag{2.8}
$$

where $G(x,\xi)$ is a Green function given by

$$
G(x,\xi) = \begin{cases} \dfrac{1}{(2\mu_*)^3}(g_1(\xi)g_2(x) + g_3(\xi)g_4(x)) & \text{for} \quad 0 \le x \le \xi, \\ \dfrac{1}{(2\mu_*)^3}(g_1(x)g_2(\xi) + g_3(x)g_4(\xi)) & \text{for} \quad \xi \quad x \le L. \end{cases} \tag{2.9}
$$

Here the functions g_1, g_2, g_3, g_4, and constants K_0, K_1, K_2, μ_* are given by

$$
\begin{aligned}
g_1(\zeta) &= \cos\mu_*\zeta \sinh\mu_*\zeta - \sin\mu_*\zeta \cosh\mu_*\zeta, \\
g_2(\zeta) &= e^{\mu_*\zeta}\cos\mu_*\zeta - \frac{K_1}{K_0}\cos\mu_*\zeta \sinh\mu_*\zeta + \frac{K_2}{K_0}\sin\mu_*\zeta \cosh\mu_*\zeta, \\
g_3(\zeta) &= \cos\mu_*\zeta \sinh\mu_*\zeta + \sin\mu_*\zeta \cosh\mu_*\zeta, \\
g_4(\zeta) &= -e^{\mu_*\zeta}\sin\mu_*\zeta + \frac{K_1}{K_0}\sin\mu_*\zeta \cosh\mu_*\zeta + \frac{K_2}{K_0}\cos\mu_*\zeta \sinh\mu_*\zeta, \\
K_0 &= 2\cos^2\mu_*L \sinh^2\mu_*L + 2\sin^2\mu_*L \cosh^2\mu_*L, \\
K_1 &= \frac{e^{2\mu_*L} - \cos 2\mu_*L}{2}, \quad K_2 = -\frac{\sin 2\mu_*L}{2}, \quad \mu_* = \frac{\mu^{1/4}}{\sqrt{2}}.
\end{aligned}
$$

By virtue of (2.8) and (2.9), we see that the solution of (2.7) satisfies

$$
\|\varphi\|_{W_p^4(0,L)} \le C\|f\|_{L^p(0,L)}. \tag{2.10}
$$

Using the a priori estimate (2.10), we show that the operator $A(d_0)$ generates an analytic semigroup on $L^p(0,L)$. Then, along the same line as in [10], we obtain the following:

Lemma 2.1. *Let Γ_0 be a smooth curve satisfying* (2.1). *Then there exists a constant $T > 0$ such that the problem* (2.6) *has a unique smooth solution for $0 \le t \quad T$.*

Lemma 2.1 implies the existence of unique solution of (2.6) for a short time:

Theorem 2.1. *Let $\Gamma_0(x)$ be a smooth curve satisfying* (2.1). *Then there exist a constant $T > 0$ and a smooth curve $\gamma(x,t)$ such that $\gamma(x,t)$ is a unique classical solution of the problem* (CSS) *for $0 \le t \quad T$.*

2.2 Long time existence

Next we shall prove a long time existence of solution to (CSS). Let us set

$$F^\lambda = 2\partial_s^2\kappa + \kappa^3 - \lambda^2\kappa.$$

Then the gradient flow (1.2) is written as

$$\partial_t\gamma = -F^\lambda \nu.$$

Since the arc length parameter s depends on time t, the following rule holds:

Lemma 2.2. *Under* (1.2)*, the following commutation rule holds*:

$$\partial_t\partial_s = \partial_s\partial_t - \kappa F^\lambda \partial_s.$$

Lemma 2.2 gives us the following:

Lemma 2.3. *Let $\gamma(x,t)$ satisfy* (1.2)*. Then the curvature $\kappa(x,t)$ of $\gamma(x,t)$ satisfies*

$$\begin{aligned}\partial_t\kappa &= -\partial_s^2F^\lambda - \kappa^2F^\lambda \\ &= -2\partial_s^4\kappa - 5\kappa^2\partial_s^2\kappa + \lambda^2\partial_s^2\kappa - 6\kappa(\partial_s\kappa)^2 - \kappa^5 + \lambda^2\kappa^3.\end{aligned} \tag{2.11}$$

Furthermore, the line element ds of $\gamma(x,t)$ satisfies

$$\partial_t ds = \kappa F^\lambda ds = (2\kappa\partial_s^2\kappa + \kappa^4 - \lambda^2\kappa^2)ds. \tag{2.12}$$

Here we introduce the following notation:

Definition 2.1. ([2]) *Let $\mathfrak{q}^r(\partial_s^l\kappa)$ be a symbol of a polynomial as follows*:

$$\mathfrak{q}^r(\partial_s^l\kappa) = \sum_m C_m \prod_{i=1}^{N_m} \partial_s^{c_{m_i}}\kappa$$

with all the c_{m_i} less than or equal to l and

$$\sum_{i=1}^{N_m}(c_{m_i}+1) = r$$

for every m, where C_m are constant coefficients.

By virtue of Lemmas 2.2 and 2.3, we have

Lemma 2.4. *For any $j \in \mathbb{N}$, the following formula holds*:

$$\partial_t\partial_s^j\kappa = -2\partial_s^{j+4}\kappa - 5\kappa^2\partial_s^{j+2}\kappa + \lambda^2\partial_s^{j+2}\kappa + \lambda^2\mathfrak{q}^{j+3}(\partial_s^j\kappa) + \mathfrak{q}^{j+5}(\partial_s^{j+1}\kappa). \tag{2.13}$$

Proof. The case $j = 0$ in (2.13) has been already proved in Lemma 2.3, where $\mathfrak{q}^5(\partial_s\kappa) = -6\kappa(\partial_s\kappa)^2 - \kappa^5$ and $\mathfrak{q}^3(\kappa) = \kappa^3$. Next suppose that the formula (2.13) holds for $j-1$. Then we have

$$
\begin{aligned}
\partial_t\partial_s^j\kappa &= \partial_s\partial_t\partial_s^{j-1}\kappa - \kappa F^\lambda \partial_s^j\kappa \\
&= \partial_s\{-2\partial_s^{j+3}\kappa - 5\kappa^2\partial_s^{j+1}\kappa + \lambda^2\partial_s^{j+1}\kappa + \lambda^2\mathfrak{q}^{j+2}(\partial_s^{j-1}\kappa) + \mathfrak{q}^{j+4}(\partial_s^j\kappa)\} \\
&\quad - \kappa(2\partial_s^2\kappa + \kappa^3 - \lambda\kappa^2)\partial_s^j\kappa \\
&= -2\partial_s^{j+4}\kappa - 5\kappa^2\partial_s^{j+2}\kappa + \lambda^2\partial_s^{j+2}\kappa + \lambda^2\mathfrak{q}^{j+3}(\partial_s^j\kappa) + \mathfrak{q}^{j+5}(\partial_s^{j+1}\kappa).
\end{aligned}
$$

We complete the proof. □

From the boundary condition of (CSS), we see that the curvature κ satisfies the following:

Lemma 2.5. *Let $\kappa(x,t)$ be the curvature of $\gamma(x,t)$ satisfying* (CSS). *Then, for any $m \in \mathbb{N}$, it holds that*

$$
\partial_s^{2m}\kappa(0,t) = \partial_s^{2m}\kappa(L,t) = 0. \tag{2.14}
$$

Proof. First we show the case where $m = 1, 2$. Differentiating the boundary condition $\gamma(0,t) = (0,0)$ and $\gamma(L,t) = (R,0)$ with respect to t, we have $\partial_t\gamma(0,t) = \partial_t\gamma(L,t) = 0$. From $\kappa(0,t) = \kappa(L,t) = 0$ and the equation (1.2), we see that $\partial_s^2\kappa(0,t) = \partial_s^2\kappa(L,t) = 0$. Since $\partial_t\kappa(0,t) = \partial_t\kappa(L,t) = 0$, the equation (2.11) yields $\partial_s^4\kappa(0,t) = \partial_s^4\kappa(L,t) = 0$.

Next, suppose that $\partial_s^{2n}\kappa(0,t) = \partial_s^{2n}\kappa(L,t) = 0$ holds for any natural numbers $n \leq m$. Lemma 2.4 gives us

$$
\partial_t\partial_s^{2m-2}\kappa = -2\partial_s^{2m+2}\kappa - 5\kappa^2\partial_s^{2m}\kappa + \lambda^2\partial_s^{2m}\kappa + \lambda^2\mathfrak{q}^{2m+1}(\partial_s^{2m-2}\kappa) + \mathfrak{q}^{2m+3}(\partial_s^{2m-1}\kappa).
$$

Since any monomials of $\mathfrak{q}^{2m+1}(\partial_s^{2m-2}\kappa)$ and $\mathfrak{q}^{2m+3}(\partial_s^{2m-1}\kappa)$ contain at least one of the terms $\partial_s^{2l}\kappa$ ($l = 0, 1, 2, \cdots, m-1$), we obtain $\partial_s^{2m+2}\kappa(0,t) = \partial_s^{2m+2}\kappa(L,t) = 0$. □

Let us define L^p norm with respect to the arc length parameter of γ. For a function $f(s)$ defined on γ, we write

$$
\|f\|_{L_s^p} = \left\{\int_\gamma |f(s)|^p\, ds\right\}^{\frac{1}{p}}.
$$

Similarly we define

$$
\|f\|_{L_s^\infty} = \sup_{s\in[0,\mathscr{L}(\gamma)]} |f(s)|,
$$

where $\mathscr{L}(\gamma)$ denotes the length of γ. Here we prepare the following interpolation inequalities:

Lemma 2.6. *Let $\gamma(x,t)$ be a solution of* (CSS). *Let $u(x,t)$ be a function defined on γ and satisfy*

$$
\partial_s^{2m}u(0,t) = \partial_s^{2m}u(L,t) = 0
$$

for any $m \in \mathbb{N}$. Then, for integers $0 \leq p \quad q \quad r$, it holds that

$$
\|\partial_s^q u\|_{L_s^2} \leq \|\partial_s^p u\|_{L_s^2}^{\frac{r-q}{r-p}} \|\partial_s^r u\|_{L_s^2}^{\frac{q-p}{r-p}}. \tag{2.15}
$$

Moreover, for integers $0 \leq p \leq q \quad r$*, it holds that*

$$\|\partial_s^q u\|_{L_s^\infty} \leq \sqrt{2}\,\|\partial_s^p u\|_{L_s^2}^{\frac{2(r-q)-1}{2(r-p)}}\,\|\partial_s^r u\|_{L_s^2}^{\frac{2(q-p)+1}{2(r-p)}}. \tag{2.16}$$

Proof. By the boundary condition of u, for any positive integer n, we have

$$\|\partial_s^n u\|_{L_s^2}^2 = \int_\gamma (\partial_s^n u)^2\,ds = -\int_\gamma \partial_s^{n-1} u \cdot \partial_s^{n+1} u\,ds \leq \|\partial_s^{n-1} u\|_{L_s^2}\,\|\partial_s^{n+1} u\|_{L_s^2}.$$

This implies that $\log \|\partial_s^n u\|_{L_s^2}$ is concave with respect to $n > 0$. Thus we obtain the inequality (2.15).

Next we turn to (2.16). Since it holds that $\partial_s^{2m} u(0) = \partial_s^{2m} u(L) = 0$ for any $m \in \mathbb{N}$, the intermediate theorem implies that there exists at least one point $0 \quad \xi \quad L$ such that $\partial_s^{2m+1} u = 0$ at $x = \xi$. Hence, for each non-negative integer n, there exists a point $0 \leq \xi_* \leq L$ such that $\partial_s^n u = 0$ at $x = \xi_*$. Then we obtain

$$\|\partial_s^n u\|_{L_s^\infty} \leq \sqrt{2}\,\|\partial_s^n u\|_{L_s^2}^{\frac{1}{2}}\,\|\partial_s^{n+1} u\|_{L_s^2}^{\frac{1}{2}}. \tag{2.17}$$

Combining (2.15) with (2.17), we obtain (2.16). □

By virtue of Lemma 2.5, we are able to apply Lemma 2.6 to $\partial_s^n \kappa$ for any non-negative integer n. Making use of boundedness of energy functional at $\gamma = \Gamma_0$, we derive an estimate for $\|\kappa\|_{L_s^2}$.

Lemma 2.7. *Let* γ *be a solution of* (CSS)*. Then the curvature* κ *satisfies*

$$\|\kappa\|_{L_s^2}^2 \leq \|k_0\|_{L^2(0,L)}^2 + \lambda^2 (\mathscr{L}(\Gamma_0) - R). \tag{2.18}$$

Proof. Since the equation in (CSS) is the steepest descent flow for $E(\gamma) = \|\kappa\|_{L_s^2}^2 + \lambda^2 \mathscr{L}(\gamma)$, we have

$$\|\kappa\|_{L_s^2}^2 + \lambda^2 \mathscr{L}(\gamma) \leq \|k_0\|_{L^2(0,L)}^2 + \lambda^2 \mathscr{L}(\Gamma_0).$$

Since it is clear that $\mathscr{L}(\gamma) \geq R$, we obtain (2.18). □

We shall prove a long time existence of solution to (CSS) by using the energy method. To do so, we prepare the following:

Lemma 2.8. *For any non-negative integer* j*, it holds that*

$$\begin{aligned} \frac{d}{dt}\|\partial_s^j \kappa\|_{L_s^2}^2 &= -2\,\|\partial_s^{j+2}\kappa\|_{L_s^2}^2 - 2\lambda^2\,\|\partial_s^{j+1}\kappa\|_{L_s^2}^2 \\ &\quad + \lambda^2 \int_\gamma \mathfrak{q}^{2j+4}(\partial_s^j \kappa)\,ds + \int_\gamma \mathfrak{q}^{2j+6}(\partial_s^{j+1}\kappa). \end{aligned} \tag{2.19}$$

Proof. By virtue of Lemma 2.4, we have

$$\begin{aligned}\frac{d}{dt}\|\partial_s^j\kappa\|_{L_s^2}^2 &= \int_\gamma 2\partial_s^j\kappa\partial_t\partial_s^j\kappa\,ds + \int_\gamma(\partial_s^j\kappa)^2\kappa F^\lambda\,ds\\ &= \int_\gamma 2\partial_s^j\kappa\left\{-2\partial_s^{j+4}\kappa - 5\kappa^2\partial_s^{j+2}\kappa + \lambda^2\partial_s^{j+2}\kappa + \lambda^2\mathfrak{q}^{j+3}(\partial_s^j\kappa) + \mathfrak{q}^{j+5}(\partial_s^{j+1}\kappa)\right\}ds\\ &\quad + \int_\gamma \kappa\partial_s^j\kappa(2\partial_s^2\kappa + \kappa^3 - \lambda\kappa^2)\,ds.\end{aligned}$$

By integrating by parts, we get

$$\int_\gamma \kappa^2\partial_s^j\kappa\partial_s^{j+2}\kappa\,ds = -\int_\gamma\left\{2\kappa\partial_s\kappa\partial_s^j\kappa\partial_s^{j+1}\kappa + \kappa^2(\partial_s^{j+1}\kappa)^2\right\}ds.$$

Consequently we obtain (2.19). □

Using Lemmas 2.7 and 2.8, we derive the estimate for the derivative of $\|\partial_s^j\kappa\|_{L_s^2}^2$ with respect to t.

Lemma 2.9. *For any non-negative integer j, we have*

$$\frac{d}{dt}\|\partial_s^j\kappa\|_{L_s^2}^2 \le C\|\kappa\|_{L_s^2}^{4j+6} + C\|\kappa\|_{L_s^2}^{4j+10}.$$

Proof. By Lemma 2.8, it is sufficient to estimate the right-hand side of (2.19). First we focus on the term $\int_\gamma \mathfrak{q}^{2j+4}(\partial_s^j\kappa)\,ds$. By Definition 2.1, we have

$$\mathfrak{q}^{2j+4}(\partial_s^j\kappa) = \sum_m\prod_{l=1}^{N_m}\partial_s^{c_{ml}}\kappa$$

with all the c_{ml} less than or equal to j and

$$\sum_{l=1}^{N_m}(c_{ml}+1) = 2j+4$$

for every m. Hence we have

$$|\mathfrak{q}^{2j+4}(\partial_s^j\kappa)| \le \sum_m\prod_{l=1}^{N_m}|\partial_s^{c_{ml}}\kappa|.$$

Setting

$$Q_m = \prod_{l=1}^{N_m}|\partial_s^{c_{ml}}\kappa|,$$

it holds that

$$\int_\gamma|\mathfrak{q}^{2j+4}(\partial_s^j\kappa)|\,ds \le \sum_m\int_\gamma Q_m\,ds.$$

We now estimate any term Q_m by Lemma 2.6. After collecting derivatives of the same order in Q_m, we can write

$$Q_m = \prod_{i=0}^{j} |\partial_s^i \kappa|^{\alpha_{mi}} \qquad \text{with} \quad \sum_{i=0}^{j} \alpha_{mi}(i+1) = 2j+4. \tag{2.20}$$

Then

$$\int_\gamma Q_m\,ds = \int_\gamma \prod_{i=0}^{j} |\partial_s^i \kappa|^{\alpha_{mi}}\,ds \le \prod_{i=0}^{j} \left(\int_\gamma |\partial_s^i \kappa|^{\alpha_{mi}}\,ds \right)^{1/\lambda_i} \le \prod_{i=0}^{j} \|\partial_s^i \kappa\|_{L_s^{\alpha_{mi}\lambda_i}}^{\alpha_{mi}},$$

where the value λ_i are chosen as follows: $\lambda_i = 0$ if $\alpha_{mi} = 0$ (in this case the corresponding term is not present in the product) and $\lambda_i = (2j+4)/(\alpha_{mi}(i+1))$ if $\alpha_{mi} \neq 0$. Clearly, $\alpha_{mi}\lambda_i = \frac{2j+4}{i+1} \ge \frac{2j+4}{j+1} > 2$ and by the condition (2.20),

$$\sum_{i=0,\lambda_i\neq 0}^{j} \frac{1}{\lambda_i} = \sum_{i=0,\lambda_i\neq 0}^{j} \frac{\alpha_{mi}(i+1)}{2j+4} = 1.$$

Let $k_i = \alpha_{mi}\lambda_i - 2$. The fact $\alpha_{mi}\lambda_i > 2$ implies $k_i > 0$. Then we have

$$\|\partial_s^i \kappa\|_{L_s^{\alpha_{mi}\lambda_i}} = \|\partial_s^i \kappa\|_{L_s^\infty}^{k_i} \|\partial_s^i \kappa\|_{L_s^2}^{2},$$

$$\|\partial_s^i \kappa\|_{L_s^\infty}^{k_i} \le 2^{\frac{k_i}{2}} \|\partial_s^{j+1} \kappa\|_{L_s^2}^{\frac{2i+1}{2j+2}k_i} \|\kappa\|_{L_s^2}^{\frac{2j+1-2i}{2j+2}k_i},$$

$$\|\partial_s^i \kappa\|_{L_s^2}^{2} \le \|\partial_s^{j+1} \kappa\|_{L_s^2}^{\frac{2i}{j+1}} \|\kappa\|_{L_s^2}^{\frac{2j+2-2i}{j+1}}.$$

These imply

$$\|\partial_s^i \kappa\|_{L_s^{\alpha_{mi}\lambda_i}} \le 2^{\frac{k_i}{2}} \|\partial_s^{j+1} \kappa\|_{L_s^2}^{\sigma_{mi}} \|\kappa\|_{L_s^2}^{1-\sigma_{mi}}$$

with

$$\sigma_{mi} = \frac{i+\frac{1}{2} - \frac{1}{\alpha_{mi}\lambda_i}}{j+1}.$$

Multiplying together all the estimates, we see that

$$\begin{aligned} \int_\gamma Q_m\,ds &\le \prod_{i=0}^{j} 2^{\frac{k_i}{2}} \|\partial_s^{j+1} \kappa\|_{L_s^2}^{\alpha_{mi}\sigma_{mi}} \|\kappa\|_{L_s^2}^{\alpha_{mi}(1-\sigma_{mi})} \\ &\le C \|\partial_s^{j+1} \kappa\|_{L_s^2}^{\sum_{i=0}^{j}\alpha_{mi}\sigma_{mi}} \|\kappa\|_{L_s^2}^{\sum_{i=0}^{j}\alpha_{mi}(1-\sigma_{mi})}. \end{aligned} \tag{2.21}$$

Then we compute

$$\sum_{i=0}^{j} \alpha_{mi}\sigma_{mi} = \sum_{i=0}^{j} \frac{\alpha_{mi}(i+\frac{1}{2}) - \frac{1}{\lambda_i}}{j+1} = \frac{\sum_{i=0}^{j} \alpha_{mi}(i+\frac{1}{2}) - 1}{j+1}$$

and using again the rescaling condition in (2.20),

$$\sum_{i=0}^{j}\alpha_{mi}\sigma_{mi} = \frac{\sum_{i=0}^{j}\alpha_{mi}(i+1) - \frac{1}{2}\sum_{i=0}^{j}\alpha_{mi} - 1}{j+1}$$
$$= \frac{2j+4-\frac{1}{2}\sum_{i=0}^{j}\alpha_{mi} - 1}{j+1} = \frac{4j+6-\sum_{i=0}^{j}\alpha_{mi}}{2(j+1)}.$$

Since

$$\sum_{i=0}^{j}\alpha_{mi} \geq \sum_{i=0}^{j}\alpha_{mi}\frac{i+1}{j+1} = \frac{2j+4}{j+1},$$

we get

$$\sum_{i=0}^{j}\alpha_{mi}\sigma_{mi} \leq \frac{2j^2+4j+1}{(j+1)^2} = 2 - \frac{1}{(j+1)^2} \quad 2.$$

Hence, we can apply the Young inequality to the product in the last term of inequality (2.21). Then we have

$$\int_\gamma Q_m\,ds \leq \frac{\delta_m}{2}\left\|\partial_s^{j+1}\kappa\right\|_{L_s^2}^2 + C_m\left\|\kappa\right\|_{L_s^2}^{\beta} \leq \delta_m\left\|\partial_s^{j+1}\kappa\right\|_{L_s^2}^2 ds + C_m\left\|\kappa\right\|_{L_s^2}^{\beta}$$

for arbitrarily small $\delta_m > 0$ and some constant $C_m > 0$. The exponent β is given by

$$\beta = \sum_{i=0}^{j}\alpha_{mi}(1-\sigma_{mi})\frac{1}{1-\frac{\sum_{i=0}^{j}\alpha_{mi}\sigma_{mi}}{2}} = \frac{2\sum_{i=0}^{j}\alpha_{mi}(1-\sigma_{mi})}{2-\sum_{i=0}^{j}\alpha_{mi}\sigma_{mi}}$$
$$= \frac{2\sum_{i=0}^{j}\alpha_{mi} - \frac{4j+6-\sum_{i=0}^{j}\alpha_{mi}}{j+1}}{2-\frac{4j+6-\sum_{i=0}^{j}\alpha_{mi}}{2(j+1)}} = 2\frac{2(j+1)\sum_{i=0}^{j}\alpha_{mi} - 4j - 6 + \sum_{i=0}^{j}\alpha_{mi}}{4j+4-4j-6+\sum_{i=0}^{j}\alpha_{mi}}$$
$$= 2\frac{(2j+3)\sum_{i=0}^{j}\alpha_{mi} - 2(2j+3)}{\sum_{i=0}^{j}\alpha_{mi} - 2} = 2(2j+3).$$

Therefore we conclude

$$\int_\gamma Q_m\,ds \leq \delta_m\left\|\partial_s^{j+1}\kappa\right\|_{L_s^2}^2 + C_m\left\|\kappa\right\|_{L_s^2}^{4j+6}.$$

Repeating this argument for all the Q_m and choosing suitable δ_m whose sum over m is less than one, we are able to verify that there exists a constant C depending only on $j \in \mathbb{N}$ such that

$$\int_\gamma \mathfrak{q}^{2j+4}(\partial_s^j\kappa)\,ds \leq \left\|\partial_s^{j+1}\kappa\right\|_{L_s^2}^2 + C\left\|\kappa\right\|_{L_s^2}^{4j+6}.$$

Reasoning similarly for the term $\mathfrak{q}^{2j+6}(\partial_s^{j+1}\kappa)$, we obtain

$$\int_\gamma \mathfrak{q}^{2j+6}(\partial_s^{j+1}\kappa)\,ds \leq \left\|\partial_s^{j+2}\kappa\right\|_{L_s^2}^2 + C\left\|\kappa\right\|_{L_s^2}^{4j+10}.$$

Hence, from (2.19), we get

$$\begin{aligned}\partial_t \|\partial_s^j \kappa\|_{L_s^2}^2 &= -2\|\partial_s^{j+2}\kappa\|_{L_s^2}^2 - 2\lambda^2\|\partial_s^{j+1}\kappa\|_{L_s^2}^2 \\ &\quad + \lambda^2 \int_\gamma \mathfrak{q}^{2j+4}(\partial_s^j \kappa)\, ds + \int_\gamma \mathfrak{q}^{2j+6}(\partial_s^{j+1}\kappa) \\ &\le -\lambda^2\|\partial_s^{j+1}\kappa\|_{L_s^2}^2 + C\|\kappa\|_{L_s^2}^{4j+6} - \|\partial_s^{j+2}\kappa\|_{L_s^2}^2 + C\varepsilon\|\kappa\|_{L_s^2}^{4j+10} \\ &\le C\|\kappa\|_{L_s^2}^{4j+6} + C\|\kappa\|_{L_s^2}^{4j+10},\end{aligned}$$

where C depends only on j. □

Since the arc length parameter s depends on t, we need to estimate the local length of $\gamma(x,t)$.

Lemma 2.10. *Let $\gamma(x,t)$ be a solution of* (CSS) *for $0 \le t \quad T$. Then there exist positive constants C_1 and C_2 such that the inequalities*

$$\frac{1}{C_1(\Gamma_0,T)} \le |\partial_x\gamma(x,t)| \le C_1(\Gamma_0,T), \tag{2.22}$$

$$|\partial_x^m |\partial_x\gamma(x,t)|| \le C_2(\Gamma_0,T) \tag{2.23}$$

hold for any $(x,t) \in [0,L] \times [0,T]$ and integer $m \ge 1$.

We omit the proof. For the proof of Lemma 2.10, see [9]. By virtue of Lemma 2.10, we prove that the system (CSS) has a unique global solution in time.

Theorem 2.2. *Let Γ_0 be a smooth planar curve satisfying the condition* (2.1). *Then there exists a unique classical solution of* (CSS) *for any time $t > 0$.*

Proof. Suppose not, then there exists a positive constant $\tilde{T}$ such that $\gamma(x,t)$ does not extend smoothly beyond $\tilde{T}$. It follows from Lemmas 2.7 and 2.9 that

$$\|\partial_s^m\kappa\|_{L_s^2}^2 \le \|\partial_x^m k_0\|_{L^2(0,L)}^2 + C\tilde{T}$$

holds for any $0 \le t \le \tilde{T}$ and non-negative integer m. This yields that there exists a constant C such that

$$\|\partial_s^m\gamma\|_{L_s^2} \le C \tag{2.24}$$

for $t \in [0,\tilde{T}]$. Here we have

$$\partial_x^m\gamma - |\partial_x\gamma|^m\partial_s^m\gamma = P(|\partial_x\gamma|,\cdots,\partial_x^{m-1}|\partial_x\gamma|,\gamma,\cdots,\partial_s^{m-1}\gamma), \tag{2.25}$$

where P is a certain polynomial. By virtue of (2.24), (2.25), and Lemma 2.10, we see that there exists a constant C such that

$$\|\partial_x^m\gamma\|_{L^2(0,L)} \le C$$

for any $t \in [0,\tilde{T}]$ and $m \in \mathbb{N}$. Then $\gamma(x,t)$ extends smoothly beyond $\tilde{T}$ by Theorem 2.1. This is a contradiction. We complete the proof. □

3 Non-compact case

In this section, we shall prove Theorem 1.1. Let $\gamma_0(x) = (\phi(x), \psi(x)) : [0,\infty) \to \mathbb{R}^2$ be a smooth curve, and κ_0 denote the curvature. Let γ_0 satisfy the conditions (C1)–(C5). Recall that γ_0 has infinite length.

Let us fix sufficiently small $\rho > 0$. The conditions (C4) and (C5) imply that, for $\rho > 0$, there exists a constant $M > 0$ such that

$$\text{(3.1)} \qquad \sup_{x\in(M,\infty)} \big||\phi'(x)| - 1\big| \quad \rho, \qquad \sup_{x\in(M,\infty)} |\psi(x)| \quad \rho, \qquad \sup_{x\in(M,\infty)} |\psi'(x)| \quad \rho.$$

Thus we see that γ_0 close to the axis in C^1 sense as $x \to \infty$.

In order to construct an approximate sequence of solution to (SS), we define a smooth curve with finite length by using γ_0. Let $\eta_r(x) \in C_c^\infty(0,+\infty)$ be a cut-off function defined by

$$\begin{aligned} \eta_r(x) &= 1 \quad \text{for any} \quad x \in [0, r-1], \\ 0 \quad \eta_r(x) & \quad 1 \quad \text{for any} \quad x \in (r-1, r), \\ \eta_r(x) &= 0 \quad \text{for any} \quad x \in [r, +\infty). \end{aligned}$$

Using the cut-off function, we define a curve $\Gamma_{0,r} : [0,r] \to \mathbb{R}^2$ as

$$\Gamma_{0,r}(x) = (\phi(x), \eta_r(x)\psi(x)) \Big|_{x\in[0,r]}.$$

Remark that the length of $\Gamma_{0,r}$ is finite. Let $\kappa_{0,r}$ denote the curvature of $\Gamma_{0,r}$. To begin with, we show the following lemma:

Lemma 3.1. *Let $r > M$. Then $\kappa_{0,r}(x)$ is smooth and satisfies*

$$\text{(3.2)} \qquad \kappa_{0,r}(r) = 0.$$

Proof. Let $r > M$. The function $\kappa_{0,r}(x)$ is written as

$$\frac{\Re(\phi'(x), \partial_x\eta_r(x)\psi(x) + \eta_r(x)\psi'(x)) \cdot (\phi''(x), \partial_x^2\eta_r(x)\psi(x) + 2\eta_r'(x)\psi'(x) + \eta_r(x)\psi''(x))}{\left|\Gamma_{0,r}'(x)\right|^3}.$$

The expression and (3.1) imply that $\kappa_{0,r} \in C^\infty(0,r)$. Furthermore, by the definition of $\eta_r(x)$, we have $\kappa_{0,r}(r) = 0$. □

To construct an approximate sequence, we consider the following initial-boundary value problem:

$$\text{(SS}_r\text{)} \qquad \begin{cases} \partial_t\gamma = (\lambda^2\kappa - 2\partial_s^2\kappa - \kappa^3)\nu, \\ \gamma(0,t) = (0,0), \quad \gamma(r,t) = (\phi(r), 0), \quad \kappa(0,t) = \kappa(r,t) = 0, \\ \gamma(x,0) = \Gamma_{0,r}(x). \end{cases}$$

Concerning (SS$_r$), we verify the following:

Lemma 3.2. *Let $r > M$. Then there exists a unique classical solution of* (SS_r) *for any time* $t > 0$.

Proof. Lemma 3.1 and Theorem 2.2 gives us the conclusion. □

In what follows, let $\gamma_r(x,t)$ denote the solution of (SS_r), and $\kappa_r(x,t)$ be the curvature of $\gamma_r(x,t)$. We prove the existence of solution of (SS) by applying the Arzela-Ascoli theorem to $\{\gamma_r\}_{r>M}$. Indeed we obtain the following:

Theorem 3.1. *Let $\gamma_0(x)$ be a planar curve satisfying* (C1)–(C5). *Then there exists a planar curve $\gamma_\infty(x,t) : [0,\infty) \times [0,\infty) \to \mathbb{R}^2$ satisfying* (SS) *on* $[0,R) \times [0,R)$ *for any* $R > M$.

In order to prove Theorem 3.1 by using the Arzela-Ascoli theorem, we need to show that γ_r and its derivatives are uniformly bounded with respect to $r > M$. Essentially the uniform boundedness of $\|\kappa_r\|_{L^2_s}$ yields the required boundedness of γ_r. Thus it is sufficient to prove the following:

Lemma 3.3. $\sup_{r\in(M,\infty)} \|\kappa_r\|_{L^2_s} \quad \infty$.

We close this paper with the outline of proof of Lemma 3.3.

Outline of proof of Lemma 3.3. Let $r > M$. First recall that the inequality

$$\|\kappa_r\|^2_{L^2_s} \leq \|\kappa_{0,r}\|^2_{L^2_s} + \lambda^2 (\mathscr{L}(\Gamma_{0,r}) - \phi(r))$$

holds. Thus it is sufficient to estimate the right-hand side.

step 1. Using the definition of $\Gamma_{0,r}$ and the condition (C2), we verify that there exists a constant C being independent of r such that

$$\|\kappa_{0,r}\|^2_{L^2_s} \quad \|\kappa_0\|^2_{L^2(0,\infty)} + C$$

for any $r > M$.

Next we turn to the second term $\mathscr{L}(\Gamma_{0,r}) - \phi(r)$. Set $M \quad b \quad r-1$. Then we have

$$\mathscr{L}(\Gamma_{0,r}) - \phi(r) = (\mathscr{L}_1(\Gamma_{0,r}) - \phi(b)) + (\mathscr{L}_2(\Gamma_{0,r}) - (\phi(r) - \phi(b))), \tag{3.3}$$

where

$$\mathscr{L}_1(\Gamma_{0,r}) = \int_0^b |\partial_x \Gamma_{0,r}(x)|\, dx = b, \quad \mathscr{L}_2(\Gamma_{0,r}) = \int_b^r |\partial_x \Gamma_{0,r}(x)|\, dx.$$

It is clear that the first term in the right-hand side of (3.3) is bounded and independent of r. Hence we focus on the second term $\mathscr{L}_2(\Gamma_{0,r}) - (\phi(r) - \phi(b))$.

step 2. From (3.1), for any $r > M$, we see that $\Gamma_{0,r}(x)$ is written by a variation of line $(\phi(x), 0)$ on the interval $[b, r]$. By virtue of a certain variational formula for the length functional of line, we see that

$$\mathscr{L}_2(\Gamma_{0,r}) - (\phi(r) - \phi(b)) \leq C\left(\|\psi\|^2_{L^2(0,\infty)} + \|\psi'\|^2_{L^2(0,\infty)}\right). \tag{3.4}$$

Then the condition (C5) implies that the second term of right-hand side of (3.3) is uniformly bounded with respect to r.

The facts obtained from step 1 and step 2 yield that $\|\kappa_r\|_{L^2_s}$ is uniformly bounded with respect to r. Therefore we complete the proof Lemma 3.3. □

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